Student Number: | | | | | | | | |

Teacher Name: ____________________

Penrith Selective High School

2022 **HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- Reference sheets are provided with this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2–5)
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6–13)
- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

	Multiple Choice	Q11	Q12	Q13	Q14	Q15	Q16	Total
Complex	4, 5 /2	a, c /7	a /4	a, e /5		b i, ii /3	a /3	/24
Proof	1, 6 /2		c /5		c, d /7			/14
Integration	7, 9 /2	b /4	b /3	b /4		b iii /3	c /6	/22
Vectors	2, 8 /2		d /3	c, d /6		a /9		/20
Mechanics	3, 10 /2	d /4			a, b /8		b /6	/20
Total	/10	/15	/15	/15	/15	/15	/15	/100

This paper MUST NOT be removed from the examination room

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Given the statement: $x^2 = 16 \Rightarrow x = \pm 4$
Which of the following statements is its contrapositive?

A. $x = \pm 4 \Rightarrow x^2 = 16$

B. $x^2 \neq 16 \Rightarrow x \neq \pm 4$

C. $x \neq \pm 4 \Rightarrow x^2 \neq 16$

D. $x \neq \pm 4 \Leftrightarrow x^2 \neq 16$

2 What is the size of the angle between the vectors $3\underset{\sim}{i} + 6\underset{\sim}{j} - 2\underset{\sim}{k}$ and $2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$, correct to the nearest minute?

A. 67°36’

B. 79°1’

C. 100°59’

D. 112°24’

3 The displacement *x* of a particle in metres after *t* seconds is given by $x = 2 + 4\sin^2 t$.
How far will the particle travel in the first 2π seconds?

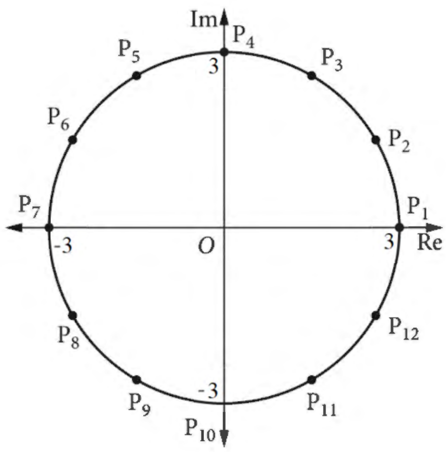
A. 16 m

B. 8 m

C. 4 m

D. 2 m

4 On the Argand diagram given, the points $P_1, P_2, P_3, \ldots, P_{12}$ are evenly spaced around the circle of radius 3. Which of the following points represent the complex numbers such that $z^3 = -27i$?



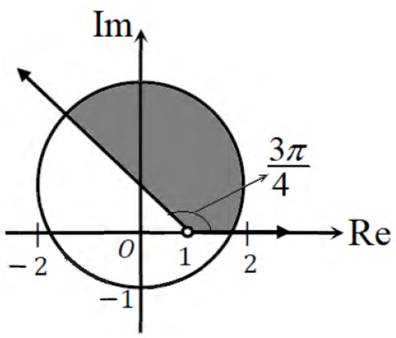
A. P_4, P_8, P_{12}

B. P_1, P_5, P_9

C. P_3, P_7, P_{11}

D. P_2, P_6, P_{10}

5 Consider the Argand Diagram shown below.



Which of the following inequalities would define the shaded area?

A. $|z-i| \le 2$ and $0 \le \arg(z-1) \le \frac{3\pi}{4}$

B. $|z-i| \le 2$ and $0 \le \arg(z+1) \le \frac{3\pi}{4}$

C. $|z-i| \ge 2$ and $0 \le \arg(z-1) \le \frac{3\pi}{4}$

D. $|z+i| \le 2$ and $0 \le \arg(z+1) \le \frac{3\pi}{4}$

6 Given the statement: "$\forall n \in Z$, $n = 4m + 3 \Rightarrow n$ can be written as a sum of two square integers"
Which of the following statements is its negation?

A. $\forall n \in Z$, $n \neq 4m + 3$ and n can be written as a sum of two square integers

B. $\exists n \in Z$, $n = 4m + 3$ and n cannot be written as a sum of two square integers

C. $\forall n \in Z$, $n = 4m + 3$ and n cannot be written as a sum of two square integers

D. $\exists n \in Z$, $n \neq 4m + 3$ and n can be written as a sum of two square integers

7 Evaluate $\int_0^{\frac{\pi}{6}} \sec^3 x \tan x \, dx$.

A. $\dfrac{8\sqrt{3}-1}{3}$

B. $\dfrac{8\sqrt{3}}{27}$

C. $\dfrac{8\sqrt{3}-9}{27}$

D. $\dfrac{8\sqrt{3}-3}{27}$

8 Which of the following represents the vector projection of $\overrightarrow{OA}$ onto $\overrightarrow{OB}$ given $A(3,-2,\ 4)$ and $B(1,\ 1,-1)$?

A. $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

B. $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

C. $\dfrac{3}{29}\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$

D. $\dfrac{3}{29}\begin{bmatrix} -3 \\ 2 \\ -4 \end{bmatrix}$

9 Which expression is equal to $\int \frac{1}{\sqrt{12-4x-x^2}}\,dx$?

A. $\sin^{-1}\left(\frac{x+2}{8}\right)+c$

B. $\frac{1}{4}\sin^{-1}(x+2)+c$

C. $\frac{1}{2}\sin^{-1}\left(\frac{x+2}{2}\right)+c$

D. $\sin^{-1}\left(\frac{x+2}{4}\right)+c$

10 A particle of mass m is moving in a straight line under the action of an applied force $F=\frac{m}{x^3}(4+6x)$. What is the equation for its velocity at any position if the particle starts from rest at $x = 1$?

A. $v=\pm\frac{1}{x}\sqrt{8x^2-6x-2}$

B. $v=\pm\frac{2}{x}\sqrt{4x^2-3x-1}$

C. $v=\frac{-4}{x^2}-\frac{12}{x}+16$

D. $v=\sqrt{\frac{-4}{x^2}-\frac{12}{x}}$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a separate Writing Booklet

(a) If $z = \sqrt{2} + i\sqrt{2}$, express each of the following in modulus–argument form:

(i) z^5 **2**

(ii) $\dfrac{1}{z}$ **1**

(b) Show that $\displaystyle\int \sec x \, dx = \log_e \left| \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \right| + C$. **4**

(c) Verify that $z = -1 + i\sqrt{3}$ is a root of the equation $z^4 - 4z^2 - 16z - 16 = 0$ and hence find the other roots. **4**

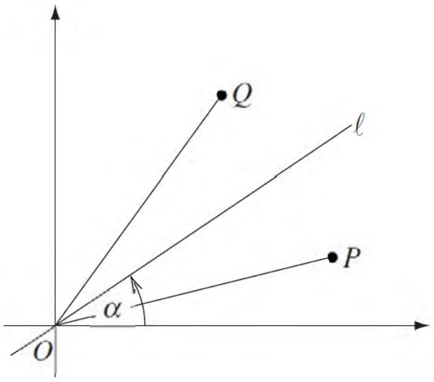
(d) (i) Prove that a particle moving according to the equation $|v| = \sqrt{-9x^2 + 12x + 32}$ is undergoing simple harmonic motion. **2**

(ii) State the period of motion. **1**

(iii) Find the range of the motion. **1**

Question 12 (15 marks) Use a separate Writing Booklet

(a) Let ℓ be the line in the complex plane that passes through the origin and makes an angle α with the positive real axis, where $0 < \alpha < \frac{\pi}{2}$.



The point P represents the complex number z_1, where $0 < \arg(z_1) < \alpha$. The point P is reflected in the line ℓ to produce the point Q, which represents the complex number z_2. Hence, $|z_1| = |z_2|$.

(i) Explain why $\arg(z_1) + \arg(z_2) = 2\alpha$. **1**

(ii) Deduce that $z_1 z_2 = |z_1|^2 (\cos 2\alpha + i \sin 2\alpha)$. **2**

(iii) Let $\alpha = \frac{\pi}{4}$ and let R be the point that represents the complex number $z_1 z_2$. **1**
Describe the locus of R as z_1 varies.

(b) Evaluate $\displaystyle\int_e^4 \frac{\ln x}{x^2}\, dx$. **3**

(c) (i) Suppose that a, b, c are real numbers. **2**
Prove that $a^4 + b^4 + c^4 \geq a^2b^2 + a^2c^2 + b^2c^2$.

(ii) Show that $a^2b^2 + a^2c^2 + b^2c^2 \geq a^2bc + b^2ac + c^2ab$. **2**

(iii) Deduce that if $a + b + c = d$, then $a^4 + b^4 + c^4 \geq abcd$. **1**

(d) Find the possible values of μ if the angle between $\underset{\sim}{p} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$ and $\underset{\sim}{q} = \begin{bmatrix} -2 \\ -4 \\ \mu \end{bmatrix}$ is $\cos^{-1} \frac{4}{21}$. **3**

Question 13 (15 marks) Use a separate Writing Booklet

(a) Use De Moivre's theorem to prove that **3**

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

(b) (i) If $\dfrac{12}{x^3+8} = \dfrac{1}{x+2} - \dfrac{x-A}{x^2-2x+4}$, find the value of A. **1**

(ii) Hence show that $\displaystyle\int \frac{12}{x^3+8}\,dx = \ln\left|\frac{x+2}{\sqrt{x^2-2x+4}}\right| + \sqrt{3}\tan^{-1}\frac{x-1}{\sqrt{3}} + C$. **3**

(c) Show that the lines $\underset{\sim}{r}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\underset{\sim}{r}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 \\ 3 \\ -3 \end{bmatrix}$ are skew. **3**

(d) Find the point(s) of intersection of the line with parametric equation **3**

$$\underset{\sim}{r} = \underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k} + \lambda\left(\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)$$

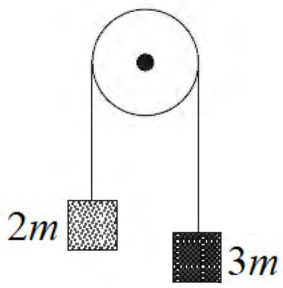
and the sphere with Cartesian equation

$$(x-1)^2 + (y-3)^2 + (z+4)^2 = 81.$$

(e) Find the two square roots of $21 - 20i$. **2**

Question 14 (15 marks) Use a separate Writing Booklet

(a) Two masses, $2m$ kg and $3m$ kg, are connected by a light inextensible string passing over a frictionless pulley as shown. **3**



Initially, the two masses are at rest. After the lighter mass has travelled x metres in an upwards direction, it is travelling at v m s^{-1}.

Prove that $v = \sqrt{\dfrac{2gx}{5}}$.

(b) A particle is moving in a straight line according to the equation

$$x = 6\cos 2t + 8\sin 2t + 5$$

where x is the displacement in metres and t is the time in seconds.

(i) Prove that the particle is moving in simple harmonic motion by showing that x satisfies an equation of the form $\ddot{x} = -n^2\left(x - c\right)$. **2**

(ii) When is the displacement of the particle zero for the first time? **3**

(c) A perfect number is a positive integer that is equal to the sum of its positive factors, excluding the number itself. Examples of perfect numbers are 6, 28 and 496.

A conjecture (an opinion or conclusion formed on the basis of incomplete information) has been proposed that if p is a perfect number then any multiple of p is also a perfect number.

(i) Use a counterexample to disprove this conjecture. **1**

(ii) Prove that if p is a perfect number then no multiple of p is a perfect number. **2**

Question 14 continues on page 10

(d) Use the principle of mathematical induction to prove that **4**

$$\sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin(2n-1)\theta = \frac{1-\cos 2n\theta}{2\sin\theta}$$

for all $n \in \mathrm{Z}^{+}$.

Question 15 (15 marks) Use a separate Writing Booklet

(a) The Tech Club in Penrith Selective High School is testing a new hybrid drone built from scratch in Room T.1.1. The drone travels in a straight line and is controlled by Anthony, using a remote ground control system.

The drone's position at any given time is expressed using coordinates (x, y, z) in kilometres, where x and y are the drone's displacement east and north of the hockey field, respectively, and z is the height of the drone above the ground.

The drone's velocity is given by the vector $\begin{bmatrix} -80 \\ -240 \\ -15 \end{bmatrix}$ km h^{-1}. Ignore air resistance.

At 10:00 a.m. Anthony detects the drone at a position 32 km east and 96 km north of the hockey field, and at a height of 8 km.

Let t be the length of time, in hours from 10:00 a.m.

(i) Write down a vector equation for the drone's displacement, $\underset{\sim}{r}$, in terms of t. **1**

(ii) Given that the drone continues to fly at the same velocity, when will it will pass directly over the hockey field and what is its height at that time? **2**

(iii) The drone continues to fly at the same velocity and descends to a height of 5 km. At what time does this happen? **2**

(iv) Calculate the direct distance, correct to one decimal place, of the drone from the hockey field at this point upon descending to a height of 5 km. **2**

(v) After descending to a height of 5 km, the drone continues to fly on the same bearing but adjusts the angle of descent so that it will land at the point $Q(0, 0, 0)$. **2**

The drone's velocity, after the adjustment of the angle of descent, is given by the vector $\begin{bmatrix} -80 \\ -240 \\ q \end{bmatrix}$ km h^{-1}. Solve for q.

Question 15 continues on page 12

Question 15 (continued)

(b) Let $z = e^{i\theta}$.

(i) Show that $z^n - \dfrac{1}{z^n} = 2i\sin(n\theta)$. **2**

(ii) Show that $\left(z - \dfrac{1}{z}\right)^5 = \left(z^5 - \dfrac{1}{z^5}\right) - 5\left(z^3 - \dfrac{1}{z^3}\right) + 10\left(z - \dfrac{1}{z}\right)$. **1**

(iii) Hence find $\int \sin^5 \theta \, d\theta$. **3**

End of Question 15

Question 16 (15 marks) Use a separate Writing Booklet

(a) It is given that the cubic equation $x^3 + bx^2 + cx + d = 0$ has a purely imaginary root. If the coefficients are all real numbers, show that $d = bc$ and $c > 0$. **3**

(b) A particle moves with acceleration function $\ddot{x} = 3x^2$. Initially $x = 1$ and $v = -\sqrt{2}$.

(i) Show that $v^2 = 2x^3$. **2**

(ii) Explain why the velocity can never be positive. **1**

(iii) Find the displacement-time function, and briefly describe the motion. **3**

(c) (i) Prove that $\sqrt{x}\left(1-\sqrt{x}\right)^{n-1} = \left(1-\sqrt{x}\right)^{n-1} - \left(1-\sqrt{x}\right)^{n}$ **1**

(ii) Let $I_n = \int_0^1 \left(1-\sqrt{x}\right)^n dx$, where $n = 1, 2, 3, \ldots$ **3**

show that $I_n = \dfrac{n}{n+2} I_{n-1}$

(iii) Hence evaluate I_{2022}. **2**

End of paper

Q1. C Q2. D Q3. A Q4. A Q5. A Q6. B Q7. C Q8. B Q9. D Q10. B

MC : 11:42

1) $x^2 = 16 \Rightarrow x = \pm 4$

contrapositive

If $x \neq \pm 4$ then $x^2 \neq 16$

$\boxed{C}$

2) $\theta = \cos^{-1}\left(\frac{3\times 2 + 6\times -2 - 2\times 1}{\sqrt{49}\sqrt{9}}\right)$

$= \cos^{-1}\left(\frac{-8}{21}\right)$

$= 112^\circ 24'$ $\boxed{D}$

3) $x = 2 + 4\sin^2 t$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2 t = 1 - \cos 2t$

$\therefore x = 2 + 2(1-\cos 2t)$

$x = 4 - 2\cos 2t$

$x = 16m$ $\boxed{A}$

4) P_4, P_8, P_{12} $\boxed{A}$

5) $\boxed{A}$

6) $\boxed{B}$

7) $\int^{\pi/4} \sec^3 x \tan x \, dx$

$= \cancel{\int_0^{\pi/6} \sec^2 x (\sec x \tan x) dx}$

$= \cancel{\sec x \tan x \cdot \tan x}$

Let $A = \sec x$

$\frac{dA}{dx} = \sec x \, dx$

$= \int_{2/\sqrt{3}}^{\frac{2}{\sqrt{3}}} \sec^2 x (\sec x \tan x) dx$

$= \int_{2/\sqrt{3}}^{1} A^2 dA$

$= \left[\frac{A^3}{3}\right]_1^1 = \frac{1}{3}\left[\frac{8}{3\sqrt{3}} - 1\right]$

$= \frac{8 - 3\sqrt{3}}{9\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{8\sqrt{3} - 9}{27}$

$\boxed{C}$

8) $\text{proj}_{\underset{\sim}{b}} \underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} (\underset{\sim}{b})$

$= \frac{3-2-4}{\sqrt{3}\times\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$= -\frac{3}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$= -1\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ $\boxed{B}$

9) $\displaystyle\int \frac{1}{\sqrt{12-4x-x^2}}\,dx$

$$= \int \frac{1}{\sqrt{16-(x^2+4x+4)}}\,dx$$

$$= \int \frac{1}{\sqrt{16-(x+2)^2}}$$

$$= \sin^{-1}\left(\frac{x+2}{4}\right) + c$$

$\boxed{D}$

10) $F = ma$

$$\ddot{x} = \frac{1}{x^3}(4+6x)$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{4}{x^3} + \frac{6}{x^2}$$

$$\frac{1}{2}v^2 = \int \left(4x^{-3} + 6x^{-2}\right)dx + c.$$

$$\frac{1}{2}v^2 = \frac{4x^{-2}}{-2} + \frac{6x^{-1}}{-1} + c$$

at $t=0$, $x=1$, $v=0$

$$0 = -2 - 6 + c$$

$$c = 8$$

$$\frac{1}{2}v^2 = -2x^{-2} - 6x^{-1} + 8$$

$$\Rightarrow v^2 = -4x^{-2} - 12x^{-1} + 16$$

$$v^2 = \frac{16x^2 - 12x - 4}{x^2}$$

$$v = \pm\sqrt{\frac{16x^2-12x-4}{x^2}}$$

$$v = \pm \frac{2}{x}\sqrt{4x^2-3x-1}$$

$\boxed{B}$

Question 11

(a) If $z = \sqrt{2} + i\sqrt{2}$, express each of the following in modulus–argument form:

(i) z^5 **2**

(ii) $\frac{1}{z}$ **1**

a) $z = \sqrt{2} + i\sqrt{2}$

$= 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$= 2\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = 2\text{cis}\frac{\pi}{4}$

i) $z^5 = 2^5 \text{cis}\frac{5\pi}{4}$

$= 32\text{cis}\frac{5\pi}{4} = 32\text{cis}\,-\frac{3\pi}{4}$ ✓✓

Common error:
-reference angle
students loss 1 mark.

ii) $\frac{1}{z} = \frac{1}{2\text{cis}\frac{\pi}{4}} = \frac{1}{2}\text{cis}\left(\frac{-\pi}{4}\right)$ ✓

(b) Show that $\int \sec x \, dx = \log_e\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$. **3**

b) $\int \sec x \, dx = \int \frac{\sec x \;(\sec x + \tan x)}{(\sec x + \tan x)} dx$

$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

since $\frac{d}{dx}(\sec x) = \tan x \sec x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$= \log|\sec x + \tan x| + c$ ✓✓

Question was poorly done.

$$\text{LHS} = \ln|\sec x + \tan x| + c$$
$$= \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + c$$

let $t = \tan\frac{x}{2}$

$\sin x = \dfrac{2t}{1+t^2}$ $\qquad \cos x = \dfrac{1-t^2}{1+t^2}$

$$= \ln\left|\frac{1+\sin x}{\cos x}\right| + c$$
$$= \ln\left|\frac{1+\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}}\right| + c$$
$$= \ln\left|\frac{(1+t)^2}{1-t^2}\right| + c$$
$$= \ln\left|\frac{(1+t)^2}{(1-t)(1+t)}\right| + c$$
$$= \ln\left|\frac{(1+t)}{(1-t)}\right| + c$$
$$= \ln\left|\frac{\tan\frac{\pi}{4} + \tan(\frac{x}{2})}{1 - \tan(\frac{x}{2})(\tan\frac{\pi}{4})}\right| + c \qquad \text{since } \tan\frac{\pi}{4} = 1$$
$$= \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + c \qquad \text{sum of angles.}$$

$$\int \sec x\,dx = \int \frac{1+t^2}{1-t^2} \times \frac{2dt}{1+t^2} = 2\int \frac{dt}{1-t^2}$$
$$= \int \left(\frac{1}{1+t} + \frac{1}{1-t}\right)dt = \log_e\left|\frac{1+t}{1-t}\right| + C$$
$$= \log_e\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C$$
$$= \log_e\left|\frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\tan\frac{x}{2}}\right| + C$$
$$= \log_e\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$$

Provided solutions

(c) Verify that $z = -1 + i\sqrt{3}$ is a root of the equation $z^4 - 4z^2 - 16z - 16 = 0$ and hence find the other roots. **4**

c) $z = -1 + i\sqrt{3} = 2\,\text{cis}\,\frac{2\pi}{3}$ sub into $z^4 - 4z^2 - 16z - 16 = 0$

$$\left(2\,\text{cis}\,\frac{2\pi}{3}\right)^4 - 4\left(2\,\text{cis}\,\frac{2\pi}{3}\right)^2 - 16\left(2\,\text{cis}\,\frac{2\pi}{3}\right) - 16 = 0$$

$$2^4\,\text{cis}\,\frac{8\pi}{3} - 4\left(2^2\,\text{cis}\,\frac{4\pi}{3}\right) - 32\,\text{cis}\,\frac{2\pi}{3} - 16 = 0$$

$$16\,\text{cis}\,\frac{8\pi}{3} - 16\,\text{cis}\,\frac{4\pi}{3} - 32\,\text{cis}\,\frac{2\pi}{3} - 16 = 0$$

$$16\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right) - 16\left(\frac{-1}{2} - \frac{i\sqrt{3}}{2}\right) - 32\left(\frac{-1}{2} + \frac{i\sqrt{3}}{2}\right) - 16 = 0$$

$$-8 + 8\sqrt{3}i + 8 + 8\sqrt{3}i + 16 - 16\sqrt{3}i - 16 = 0$$

$$0 = 0$$

mention roots are real and conjugate roots occur.

$\therefore z = -1 + i\sqrt{3}$ is a root of the given equation.

$\therefore z = -1 - i\sqrt{3}$ is also a root.

$$(z + 1 + i\sqrt{3})(z + 1 - i\sqrt{3}) = z^2 + z - zi\sqrt{3} + z + 1 - i\sqrt{3} + zi\sqrt{3} + i\sqrt{3} + 3$$
$$= z^2 + 2z + 4$$

$$\begin{array}{r|l}
 & z^2 - 2z - 4 \\
z^2 + 2z + 4 & z^4 + 0z^3 - 4z^2 - 16z - 16 \\
 & -(z^4 + 2z^3 + 4z^2) \\
 & 0 - 2z^3 - 8z^2 - 16z \\
 & -(-2z^3 - 4z^2 - 8z) \\
 & 0 - 4z^2 - 8z - 16 \\
 & -(4z^2 - 8z - 16) \\
 & 0 \quad 0 \quad 0
\end{array}$$

alternate method: Sum of roots product of roots

$z^2 - 2z - 4$. $z = \frac{2 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5}$

$\therefore$ other roots $z = -1 - i\sqrt{3},\ 1 \pm \sqrt{5}$

(d) (i) Prove that a particle moving according to the equation $|v| = \sqrt{-9x^2 + 12x + 32}$ is undergoing simple harmonic motion. **2**

i) $|v| = \sqrt{-9x^2 + 12x + 32}$

alternate $v^2 = n^2(a^2 - x^2)$

$\frac{1}{2}v^2 = -\frac{9}{2}x^2 + 6x + 16$ ✓

$\ddot{x} = \frac{d}{dx}\left(-\frac{9}{2}x^2 + 6x + 16\right)$ ✓

$= -9x + 6$

$= -9\left(x - \frac{2}{3}\right)$ (in the form $-n^2(x-c)$)

(ii) State the period of motion. **1**

ii) $n^2 = 9$

$n = 3$

Period $= \frac{2\pi}{3}$ ✓

(iii) Find the range of the motion. ~~**2**~~ 1 mark

iii) $\frac{1}{2}v^2 = -\frac{9}{2}x^2 + 6x + 16$

$v^2 = -9x^2 + 12x + 32$

$= 9\left(-x^2 + \frac{12}{9}x + \frac{32}{9}\right)$

$= 9\left(-x^2 + \frac{4}{3}x - \frac{4}{9} + \frac{32}{9} + \frac{4}{9}\right)$

$= 9\left(-\left(x - \frac{2}{3}\right)^2 + 4\right)$

$= 9\left(4 - \left(x - \frac{2}{3}\right)^2\right)$

$\therefore a^2 = 4,\ a = 2.$

range: ± 2

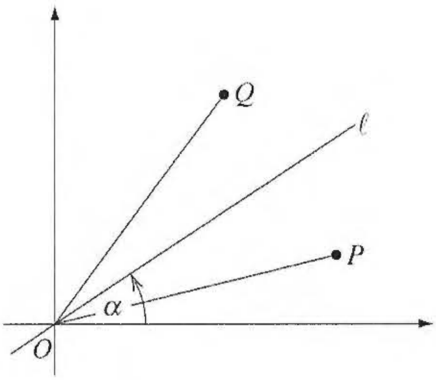
2 units away from the centre of motion.

$\left(\frac{8}{3} \text{ and } -\frac{4}{3}\right)$

centre of motion: $+\frac{2}{3} \pm 2.$

Question 12 (15 marks) Use a separate Writing Booklet

(a) Let ℓ be the line in the complex plane that passes through the origin and makes an angle α with the positive real axis, where $0 < \alpha < \frac{\pi}{2}$.

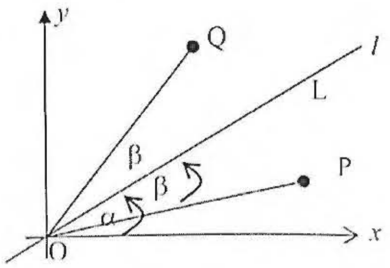


The point P represents the complex number z_1, where $0 < \arg(z_1) < \alpha$. The point P is reflected in the line ℓ to produce the point Q, which represents the complex number z_2. Hence, $|z_1| = |z_2|$.

(i) Explain why $\arg(z_1) + \arg(z_2) = 2\alpha$. **1**

Solution

(i)



Let $\angle POL$ be β, then by symmetry,
$\angle LOQ = \beta$, and
$\arg(z_2) = \alpha + \beta$
$\arg(z_1) = \alpha - \beta$
$\therefore \arg(z_1) + \arg(z_2) = 2\alpha$

- good attempt,
- students need to strengthen their reasoning

(ii) Deduce that $z_1 z_2 = |z_1|^2 (\cos 2\alpha + i \sin 2\alpha)$. **2**

Solution

(ii)

$z_1 = |z_1| cis(\alpha - \beta)$ *and* $z_2 = |z_1| cis(\alpha + \beta)$

$\therefore z_1 z_2 = |z_1|^2 cis(\alpha - \beta + \alpha + \beta)$
$= |z_1|^2 cis(2\alpha)$

- good attempt
- students need to write using part (i) etc.

(iii) Let $\alpha = \frac{\pi}{4}$ and let R be the point that represents the complex number $z_1 z_2$. **1**
Describe the locus of R as z_1 varies.

Solution

(iii) $R = |z_1|^2 cis(2\alpha) = |z_1|^2 cis\frac{\pi}{2} = i|z_1|^2$

$\therefore$ R is purely imaginary and $|z_1|^2 > 0$. Hence the locus of R is the **y – axis, $y > 0$.**

- poor attempt - as many student fail to recognise that i indicates imaginary axis

(b) Evaluate $\int_e^4 \frac{\ln x}{x^2}\, dx$. **3**

Solution

If u, v are functions of x, then integrating by parts,

$$\int u.\frac{dv}{dx}.dx = uv - \int v.\frac{du}{dx}.dx$$

$$\int_e^4 \frac{\ln x}{x^2}\,dx,\ i.e.\ \int_e^4 \ln x.x^{-2}dx$$

$$= \int_e^4 \ln x.\frac{d}{dx}\left(-x^{-1}\right)dx \quad where\ u = \ln x,\quad v = -x^{-1}$$

$$= \left[\ln x. - x^{-1}\right]_e^4 - \int_e^4 -x^{-1}.\frac{d}{dx}(\ln x)dx$$

$$= \left[-\frac{1}{x}\ln x\right]_e^4 + \int_e^4 \frac{1}{x}.\frac{1}{x}.dx$$

$$= \left\{-\frac{1}{4}\ln 4 + \frac{1}{e}\ln e\right\} + \int_e^4 x^{-2}dx$$

$$= -\frac{1}{4}\ln 4 + \frac{1}{e} + \left[-x^{-1}\right]_e^4$$

$$= -\frac{1}{4}\ln 4 + \frac{1}{e} + \left[-\frac{1}{4} + \frac{1}{e}\right]$$

$$= \frac{2}{e} - \frac{1}{4} - \frac{1}{4}\ln 4$$

- overall students did this question well
- still issue with +/− signs

(c) (i) Suppose that a, b, c are real numbers. **2**

Prove that $a^4 + b^4 + c^4 \geq a^2b^2 + a^2c^2 + b^2c^2$.

Solution

Using AMGM: Needs to be proven $\frac{x+y}{2} \geq \sqrt{xy}$

(ii) Using the result in (i), then

$$\frac{a^4 + b^4}{2} \geq \sqrt{a^4b^4} = a^2b^2$$

$$\frac{b^4 + c^4}{2} \geq \sqrt{b^4c^4} = b^2c^2$$

$$\frac{c^4 + a^4}{2} \geq \sqrt{c^4a^4} = c^2a^2$$

By addition,

$$\frac{\left(a^4 + b^4\right) + \left(b^4 + c^4\right) + \left(c^4 + a^4\right)}{2} \geq a^2b^2 + b^2c^2 + c^2a^2$$

i.e. $\frac{2\left(a^4 + b^4 + c^4\right)}{2} \geq a^2b^2 + b^2c^2 + c^2a^2$

i.e. $a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2$

well done.

(ii) Show that $a^2b^2+a^2c^2+b^2c^2 \ge a^2bc+b^2ac+c^2ab$. **2**

Solution

(iii) Using the result in (i) again, then

$$\frac{a^2b^2+a^2c^2}{2}=\frac{a^2\left(b^2+c^2\right)}{2}\ge\sqrt{a^2b^2.a^2c^2}=a^2bc$$

$$\frac{b^2c^2+b^2a^2}{2}=\frac{b^2\left(c^2+a^2\right)}{2}\ge\sqrt{b^2c^2.b^2a^2}=b^2ca$$

$$\frac{c^2a^2+c^2b^2}{2}=\frac{c^2\left(a^2+b^2\right)}{2}\ge\sqrt{c^2a^2.c^2b^2}=c^2ab$$

– well done

– Very few did not know where to start from.

By addition,

$$\frac{a^2\left(b^2+c^2\right)+b^2\left(c^2+a^2\right)+c^2\left(a^2+b^2\right)}{2}$$

$$\ge a^2bc+b^2ca+c^2ab$$

i.e. $\dfrac{2\left(a^2b^2+b^2c^2+c^2a^2\right)}{2}\ge a^2bc+b^2ca+c^2ab$.

i.e. $a^2b^2+b^2c^2+c^2a^2\ge a^2bc+b^2ca+c^2ab$.

(iii) Deduce that if $a+b+c=d$, then $a^4+b^4+c^4\ge abcd$. **1**

Solution

(iv) From (ii), (iii)

$a^4+b^4+c^4\ge a^2b^2+b^2c^2+c^2a^2$ and

$a^2b^2+b^2c^2+c^2a^2\ge a^2bc+b^2ca+c^2ab$

Thus $a^4+b^4+c^4\ge a^2bc+b^2ca+c^2ab$

$= abc\,(a+b+c)$, factorising

$= abcd$, since $a+b+c=d$

– well done.

$a^4+b^4+c^4\ge abcd$.

(d) Find the possible values of μ if the angle between $\underset{\sim}{p}=\begin{bmatrix}6\\-2\\3\end{bmatrix}$ and $\underset{\sim}{q}=\begin{bmatrix}-2\\-4\\\mu\end{bmatrix}$ is $\cos^{-1}\dfrac{4}{21}$. **3**

Solution 4 and $\boxed{-44/65}$ → Only a few could show that $\frac{-44}{65}$ is not a valid answer.

overall, calculation errors were there

Adding columnwise.

7

$a^2b^2 + b^2c^2 + c^2a^2 \geq a^2bc + b^2ac + c^2ab$

(iii) using part(i)

$a^4 + b^4 + c^2 \geq a^2b^2 + b^2c^2 + c^2a^2$ (using part(ii)

$\geq a^2bc + b^2ac + c^2ab$

$a^2bc + b^2ac + c^2ab = abc\,(a+b+c)$

$= abcd$, it is given $a+b+c = d$.

$\therefore \quad a^4 + b^4 + c^4 \geq abcd$.

d) $\underset{\sim}{p} \cdot \underset{\sim}{q} = |\underset{\sim}{p}|\,|\underset{\sim}{q}| \cos\theta$, where θ is the angle between $\underset{\sim}{p}$ and $\underset{\sim}{q}$.

$\dfrac{6\times -2 + -2\times -4 + 3\times \mu}{|\underset{\sim}{p}|\,|\underset{\sim}{q}|} = \cos\theta$,

$|\underset{\sim}{p}| = \sqrt{6^2+2^2+3^2}$, $|\underset{\sim}{q}| = \sqrt{2^2+4^2+\mu^2}$

$= \sqrt{49}$ $\therefore |\underset{\sim}{q}| = \sqrt{20+\mu^2}$

$\therefore \quad |\underset{\sim}{p}| = 7$

$\therefore \cos\theta = \dfrac{3\mu - 4}{7\sqrt{20+\mu^2}}$, Also, $\cos\theta = \dfrac{4}{21}$

$\dfrac{4}{21} = \dfrac{3\mu - 4}{7\sqrt{20+\mu^2}}$

solving the equation, (8)

$$\frac{3\mu-4}{\sqrt{20+\mu^2}} = \frac{4}{3}$$

$$9\mu - 12 = 4\sqrt{20+\mu^2}$$

squaring both sides,

$$81\mu^2 + 144 - 2\times 9\times 12\mu = 16(20+\mu^2)$$

$$65\mu^2 - 216\mu - 176 = 0$$

$$\Rightarrow \mu = \frac{216 \pm \sqrt{(216)^2 - 4(65)(-176)}}{2\times 65}$$

$$= \frac{216 \pm 304}{130}$$

$$\therefore \mu = \frac{\cancel{520}^{\,4}}{\cancel{130}} \text{ or } \frac{\cancel{-88}^{\,-44}}{\cancel{130}\,65}$$

the possible values of μ are 4 and $\frac{-44}{65}$.

<u>validity of μ:</u>

sub $\mu = -\frac{44}{65}$, $\underset{\sim}{q} = \begin{pmatrix} -2 \\ -4 \\ -\frac{44}{65} \end{pmatrix}$

$\underset{\sim}{p} \cdot \underset{\sim}{q} = \ldots$

Not valid.

Only valid value of $\mu = 4$.

Question 13

(a) Use de Moivre's theorem to prove that **3**

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

a) Binomial expansion

$$(\cos\theta + i\sin\theta)^4 = \underbrace{\cos^4\theta} + 4i\cos^3\theta\sin\theta \underbrace{- 6\cos^2\theta\sin^2\theta} - 4i\cos\theta\sin^3\theta + \underbrace{\sin^4\theta}$$

de Moivre's theorem

$$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$$

Equate real components

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ &= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 \\ &= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + (1 - 2\cos^2\theta + \cos^4\theta) \\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$

(b) (i) If $\dfrac{12}{x^3+8} = \dfrac{1}{x+2} - \dfrac{x-A}{x^2-2x+4}$, find the value of A. **1**

Sub $x=0$

$$\frac{12}{0+8} = \frac{1}{0+2} - \frac{0-A}{0-0+4}$$

$$\frac{12}{8} = \frac{1}{2} + \frac{A}{4}$$

$$\frac{A}{4} = 1$$

$$\therefore A = 4$$

CFE if $A \neq 4$.

(ii) Hence show that $\int \frac{12}{x^3+8}\,dx = \ln\left|\frac{x+2}{\sqrt{x^2-2x+4}}\right| + \sqrt{3}\tan^{-1}\frac{x-1}{\sqrt{3}} + C$.

ii) From part i)

$$\int \frac{12}{x^3+8}\,dx = \int\left(\frac{1}{x+2} - \frac{x-4}{x^2-2x+4}\right)dx$$

$$= \int \frac{1}{x+2}\,dx - \int \frac{x-4}{x^2-2x+4}\,dx$$

$$= \ln(x+2) - \int \frac{x-1}{x^2-2x+4}\,dx + \int \frac{3}{x^2-2x+4}\,dx$$

$$= \ln(x+2) - \frac{1}{2}\int \frac{2x-2}{x^2-2x+4}\,dx + \int \frac{3}{x^2-2x+4}\,dx$$

$$= \ln(x+2) - \frac{1}{2}\ln(x^2-2x+4) + \int \frac{3}{(x^2-2x+1)+3}\,dx$$

$$= \ln(x+2) - \ln(x^2-2x+4)^{1/2} + \int \frac{3}{(x-1)^2+3}\,dx$$

$$= \ln(x+2) - \ln\left(\sqrt{x^2-2x+4}\right) + \sqrt{3}\int \frac{\sqrt{3}}{(x-1)^2+(\sqrt{3})^2}$$

$$= \ln\left(\frac{x+2}{\sqrt{x^2-2x+4}}\right) + \sqrt{3}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + C$$

- No marks for students applying part (i)

- Students struggled to break $-\int \frac{x-4}{x^2-2x+4}$ into $-\int \frac{x-1}{x^2-2x+4} + \int \frac{3}{x^2-2x+4}$

  If they couldn't do this, it was harder to score the next two marks.

(c) Show that the lines $\underset{\sim}{r}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\underset{\sim}{r}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 \\ 3 \\ -3 \end{bmatrix}$ are skew. **3**

c) $b_1 = \mu b_2$

$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \mu \begin{pmatrix} -4 \\ 3 \\ -3 \end{pmatrix}$ not multiples of each other $\therefore$ not parallel.

$1 + 2\lambda_1 = 1 - 4\lambda_2 \rightarrow 2\lambda_1 = -4\lambda_2$ (1)

$0 - \lambda_1 = 1 + 3\lambda_2 \rightarrow \lambda_1 = -1 - 3\lambda_2$ (2)

$-1 + \lambda_1 = 0 - 3\lambda_2 \rightarrow \lambda_1 = 1 - 3\lambda_2$ (3)

Work with 1 and 2

Sub 2 into 1

$2(-1 - 3\lambda_2) = -4\lambda_2$

$-2 - 6\lambda_2 = -4\lambda_2$

$-2 = 2\lambda_2$

$\therefore \lambda_2 = -1$

sub $\lambda_2 = -1$ into 2

$\lambda_1 = -1 - 3(-1)$

$\therefore \lambda_1 = 2$

sub $\lambda_1 = 2$, $\lambda_2 = -1$ into 3 (check)

$2 = 1 - 3(-1)$

$2 \neq 4$

$\therefore$ The lines do not intersect.

$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} -4 \\ 3 \\ -3 \end{pmatrix} = -8 + (-3) + (-3) = -14$

$\neq 0$ $\therefore$ not perpendicular

$\therefore$ The lines must be skew.

(d) Find the point(s) of intersection of the line with parametric equation **3**

$$\underset{\sim}{r} = \underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k} + \lambda\left(\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)$$

and the sphere with Cartesian equation

$$(x-1)^2 + (y-3)^2 + (z+4)^2 = 81.$$

d) $r = i + 3j - 4k + t(i + 2j + 2k)$
$= (1+t)i + (3+2t)j + (-4+2t)k$

$\therefore x = 1+t$
$y = 3+2t$
$z = -4+2t$

sub into sphere equation.

$((1+t)-1)^2 + (3+2t-3)^2 + (-4+2t-4)^2 = 81$
$t^2 + (2t)^2 + (2t)^2 = 81$
$9t^2 = 81$
$t^2 = 9$
$t = \pm 3$

$x = 1+3 = 4$, $x = 1-3 = -2$
$y = 3+6 = 9$, $y = 3-6 = -3$
$z = -4+6 = 2$, $z = -4-6 = -10$

$(4, 9, 2)$ and $(-2, -3, -10)$ are points of intersection.

(e) Find the two square roots of $21 - 20i$. **2**

e) let $21 - 20i = (a+bi)^2$
$= a^2 + 2abi - b^2$
$= a^2 - b^2 + 2abi$

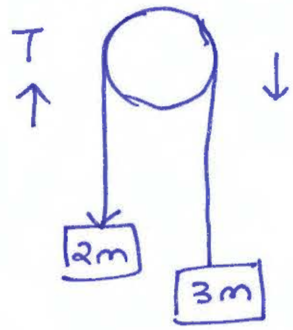
Equate real and imaginary
$a^2 - b^2 = 21$ (1)
$2ab = -20$ (2)
From (2) $a = \frac{-10}{b}$, sub into (1)

$\left(\frac{-10}{b}\right)^2 - b^2 = 21$

$\frac{100}{b^2} - b^2 = 21$
$100 - b^4 = 21b^2$
$0 = b^4 + 21b^2 - 100$
$0 = (b^2 - 4)(b^2 + 25)$
$\therefore b^2 = 4 \quad b^2 = -25$
$b = \pm 2$
when $b = 2, a = -5$ when $b = -2, a = 5$
$\therefore -5 + 2i$ and $5 - 2i$ are square roots of $21 - 20i$

Q14

a)



Net force :

$$T - 2mg = 2m\ddot{x} \quad (\cancel{\text{upward}})$$

Also, $3mg - T = 3m\ddot{x}$

adding the two equations

$$mg = 5m\ddot{x}$$

$$\ddot{x} = \frac{g}{5}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{g}{5}$$

$$\frac{1}{2}v^2 = \frac{gx}{5} + c$$

at $x = 0, v = 0 \Rightarrow c = 0$

$$\frac{1}{2}v^2 = \frac{gx}{5}$$

$$v^2 = \frac{2gx}{5}$$

$$v = \sqrt{\frac{2gx}{5}} \quad (v > 0).$$

- well done
- students did a good job writing the net force.
- Majority of students did not mention why choose $v > 0$ from $v^2 = \frac{2gx}{5}$

(b) A particle is moving in a straight line according to the equation

$$x = 6\cos 2t + 8\sin 2t + 5$$

where x is the displacement in metres and t is the time in seconds.

(i) Prove that the particle is moving in simple harmonic motion by showing that x satisfies an equation of the form $\ddot{x} = -n^2(x - c)$. **2**

Solution

(i) Prove $\ddot{x} = -n^2(x - c)$

$$x = 5 + 6\cos 2t + 8\sin 2t$$
$$\dot{x} = -12\sin 2t + 16\cos 2t$$
$$\ddot{x} = -24\cos 2t - 32\sin 2t$$
$$= -4(6\cos 2t + 8\sin 2t)$$
$$= -2^2(5 + 6\cos 2t + 8\sin 2t - 5)$$
$$= -2^2(x - 5) \quad \text{... as required}$$

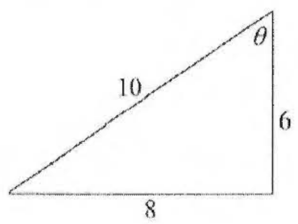
- students were able to show $\ddot{x} = -n^2(x-c)$ demonstrating their knowledge of SHM.

(ii) When is the displacement of the particle zero for the first time? **3**

Solution

(ii) Need to find t when $x = 0$ for 1st time

$$5 + 6\cos 2t + 8\sin 2t = 0$$
$$6\cos 2t + 8\sin 2t = -5$$
$$\frac{6}{10}\cos 2t + \frac{8}{10}\sin 2t = -\frac{1}{2}$$



$$\Rightarrow \cos\theta = \frac{6}{10} \text{ and } \sin\theta = \frac{8}{10}$$
$$\cos 2t\cos\theta + \sin 2t\sin\theta = -\frac{1}{2}$$
$$\cos(2t - \theta) = -\frac{1}{2}$$

Since $\cos\frac{\pi}{3} = \frac{1}{2}$ and cos is negative in the 2nd and 3rd quadrants,

$$\Rightarrow 2t - \theta = \pi - \frac{\pi}{3},\ \pi + \frac{\pi}{3}$$

We need the 1st time $x = 0$

Since $\cos\theta = \frac{6}{10}$

$$\theta = 0.9273...$$

$$\therefore\ 2t - 0.9273... = \frac{2\pi}{3}$$
$$2t = \frac{2\pi}{3} + 0.9273...$$
$$t = \frac{1}{2}\left(\frac{2\pi}{3} + 0.9273...\right)$$
$$= 1.5108...$$
$$= 1.5 \text{ seconds (1 d.p.)}$$

- some students did not use~~d~~ Auxillary angle technique
- calculation errors were visible in ~~my~~ many papers.

(c) A perfect number is a positive integer that is equal to the sum of its positive factors, excluding the number itself. Examples of perfect numbers are 6, 28 and 496.

A conjecture (an opinion or conclusion formed on the basis of incomplete information) has been proposed that if p is a perfect number then any multiple of p is also a perfect number.

(i) Use a counterexample to disprove this conjecture. **1**

Solution

6 is a perfect number and 12 is a multiple of 6.
Factors of 12 are $\{1, 2, 3, 4, 6, 12\}$.

$$1 + 2 + 3 + 4 + 6 = 16 > 12$$

This counterexample disproves the conjecture since we have a multiple of a perfect number that isn't a perfect number.

- Well done,
- Almost all students could show an example to disprove this conjecture.

(ii) Prove that if p is a perfect number then no multiple of p is a perfect number. **2**

Solution

p is a perfect number. Let the n factors of p (excluding p) in ascending order be $\{f_1, f_2, f_3, \dots, f_n\}$. Note that $f_1 = 1$ since it is a factor of all positive integers.

Let k be an integer such that $k \geq 2$. Assume that kp is a perfect number, thus

$$\begin{aligned} kp &= k(f_1 + f_2 + f_3 + \cdots + f_n) \\ &= kf_1 + kf_2 + kf_3 + \cdots + kf_n \end{aligned}$$

Note that $kf_1 > 1$ but 1 is a factor of kp so 1 must be included into the sum, thus

$$1 + kf_1 + kf_2 + kf_3 + \cdots + kf_n > kp$$

This contradicts our assumption, therefore kp is not perfect.

- Poorly done
- students did not know where to start
- students need to strengthen their ~~proofing~~ ability.

Question 14 continues on page 10

b) $\sin\theta + \sin 3\theta + \quad + \sin(2n-1)\theta = \dfrac{1-\cos 2n\theta}{2\sin\theta}$

$n \in \mathbb{Z}^+$

for $n=1$, LHS $= \sin\theta$

$$\text{RHS} = \frac{1-\cos 2\theta}{2\sin\theta}$$

$$= \frac{\cancel{\cos^2\theta} + \sin^2\theta - \cancel{\cos^2\theta} + \sin^2\theta}{2\sin\theta}$$

$$= \frac{2\sin^2\theta}{2\sin\theta}$$

$$= \sin\theta$$

LHS = RHS, result is true for $n=1$

Let the result be true for $n=k$, $k \in \mathbb{Z}^+$

$\Rightarrow \sin\theta + \sin 3\theta + - + \sin(2k-1)\theta = \dfrac{1-\cos 2k\theta}{2\sin\theta}$

To prove that the result is true for $n=k+1$

i.e. $\sin\theta + \sin 3\theta + \text{—} + \sin(2k-1)\theta + \sin(2k+1)\theta = \dfrac{1-\cos 2(k+1)\theta}{\sin\theta}$.

LHS: $\sin\theta + \sin 3\theta + - + \sin(2k-1)\theta + \sin(2k+1)\theta$

$= \dfrac{1-\cos 2k\theta}{2\sin\theta} + \sin(2k+1)\theta$ (using assumption)

$$= \frac{1-\cos 2k\theta + 2\sin\theta\sin(2k+1)\theta}{2\sin\theta}$$

$$= \frac{1-\cos 2k\theta + \cos(\theta - 2k\theta - \theta) - \cos(\theta + 2k\theta + \theta)}{2\sin\theta}$$

(using products to sums)

$= \dfrac{1-\cancel{\cos 2k\theta} + \cancel{\cos(-2k\theta)} - \cos 2(k+1)\theta}{2\sin\theta}$ as $\cos(-\theta) = \cos\theta$ even fn

$$= \frac{1-\cos 2(k+1)\theta}{2\sin\theta}$$

= RHS.

Hence using Principle of Mathematical Induction the result is true for all $n \in \mathbb{Z}^+$

Question 15

(a) The Tech Club in Penrith Selective High School is testing a new hybrid drone built from scratch in Room T.1.1. The drone travels in a straight line and is controlled by Anthony, using a remote ground control system.

The drone's position at any given time is expressed using coordinates (x, y, z) in kilometres, where x and y are the drone's displacement east and north of the hockey field, respectively, and z is the height of the drone above the ground.

The drone's velocity is given by the vector $\begin{bmatrix} -80 \\ -240 \\ -15 \end{bmatrix}$ km h^{-1}. Ignore air resistance.

At 10:00 a.m. Anthony detects the drone at a position 32 km east and 96 km north of the hockey field, and at a height of 8 km.

Let t be the length of time, in hours from 10:00 a.m.

(i) Write down a vector equation for the drone's displacement, $\underset{\sim}{r}$, in terms of t. **1**

$$\underset{\sim}{r} = \begin{pmatrix} 32 \\ 96 \\ 8 \end{pmatrix} + t\begin{pmatrix} -80 \\ -240 \\ -15 \end{pmatrix}$$

✓ 1 mark

(ii) Given that the drone continues to fly at the same velocity, when will it will pass directly over the hockey field and what is its height at that time? ~~**3**~~ 2

From part (i)

$x = 32 - 80t$ — ①

$y = 96 - 240t$ — ②

$z = 8 - 15t$ — ③

To be over the hockey field x and y equal 0.

sub $x = 0$ into ①

$0 = 32 - 80t$

$\therefore t = 0.4$

sub $t = 0.4$ into ②

$y = 96 - 240(0.4)$

$= 0$

$\therefore$ x and $y = 0$ when $t = 0.4$ ✓

Converting hours to minutes, we get

$t = 0.4$ hours

$= 0.4 \times 60$

$= 24$ mins

10am + 24mins = 10:24am.

Sub $t = 0.4$ into ③ for height.

$z = 8 - 15(0.4)$

$= 2$ km ✓

(iii) The drone contines to fly at the same velocity and descends to a height of 5 km. At what time does this happen? **2**

Sub $z=5$ (height)

$5 = 8 - 15t$

$15t = 3$

$t = 0.2$ hours

Into minutes

$t = 0.2(60)$

$= 12$ mins

$\therefore$ The drone has a height of 5km at 10:12am.

(iv) Calculate the direct distance, correct to one decimal place, of the drone from the hockey field at this point upon descending to a height of 5 km. **2**

Sub $t = 0.2$ into $\underset{\sim}{r}$ part (i).

$$\underset{\sim}{r} = \begin{pmatrix} 32 \\ 96 \\ 8 \end{pmatrix} + 0.2\begin{pmatrix} -80 \\ -240 \\ -15 \end{pmatrix}$$

$$= \begin{pmatrix} 16 \\ 48 \\ 5 \end{pmatrix}$$

Magnitude of a vector formula

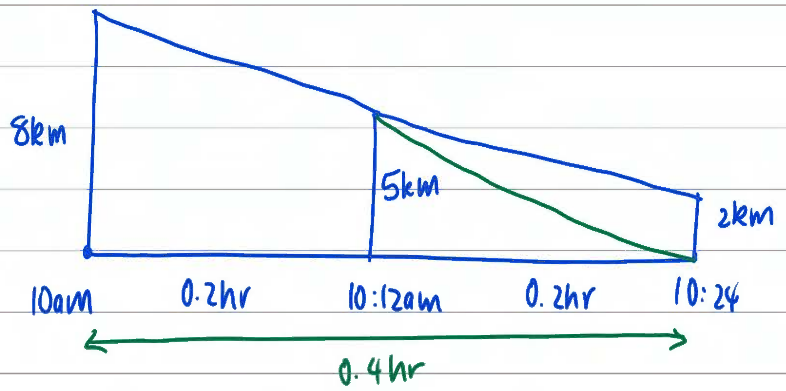
$|r| = \sqrt{16^2 + 48^2 + 5^2}$

$= 50.8$ km (1 dp).

(v) After descending to a height of 5 km, the drone continues to fly on the same bearing but adjusts the angle of descent so that it will land at the point $Q(0, 0, 0)$. **2**

The drone's velocity, after the adjustment of the angle of descent, is given by the vector $\begin{bmatrix} -80 \\ -240 \\ q \end{bmatrix}$ km h^{-1}. Solve for q.

Diagram:



$$q = -\frac{\text{height to descend}}{\text{time to land}}$$

$$= \frac{-5}{0.2}$$

$$= -25$$

Part a)

Easy question, many students scored well in the question.

Common mistake leaving time in hours when the question asked to convert it into a real time eg. 10:24am.

v) Only a few students drew a diagram to help with understanding.

Question 15 b)

Let $z = e^{i\theta}$.

(i) Show that $z^n - \frac{1}{z^n} = 2i\sin(n\theta)$. **1**

$$z^n - \frac{1}{z^n} = e^{in\theta} - \frac{1}{e^{in\theta}} \quad \text{(by De Moivre's theorem)}$$
$$= e^{in\theta} - e^{-in\theta}$$
$$= \cos(n\theta) + i\sin(n\theta) - (\cos(-n\theta) + i\sin(-n\theta))$$
$$= \cos(n\theta) + i\sin(n\theta) - \cos(n\theta) + i\sin(n\theta) \quad \text{(since cos even, sin odd)}$$
$$= 2i\sin(n\theta)$$

✓

(ii) Show that $\left(z-\frac{1}{z}\right)^5 = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$. **1**

$$\left(z-\frac{1}{z}\right)^5 = \binom{5}{0}z^5 + \binom{5}{1}z^4\left(\frac{-1}{z}\right)^1 + \binom{5}{2}z^3\left(\frac{-1}{z}\right)^2 + \binom{5}{3}z^2\left(\frac{-1}{z}\right)^3$$
$$+ \binom{5}{4}z^1\left(\frac{-1}{z}\right)^4 + \binom{5}{5}\left(\frac{-1}{z}\right)^5$$
$$= z^5 - 5z^3 + 10z - 10\left(\frac{1}{z}\right) + 5\left(\frac{1}{z^3}\right) - \frac{1}{z^5}$$
$$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

✓

Q15b) Poorly done.
Some students scored marks for part i) and ii)
But most skipped part iii)

(iii) Hence find $\int \sin^5 \theta \, d\theta$. **3**

Sub result in part (i) into identity of part (ii)

$$z^n - \frac{1}{z^n} = 2i\sin(n\theta)$$ part (i)

$$\left(z - \frac{1}{z}\right)^5 = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

iii)

$$(2i\sin\theta)^5 = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$$ ✓ 1 mark

$$32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$$

$$\therefore \sin^5\theta = \frac{1}{32i}\left(2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta\right)$$

$$= \frac{1}{16}\left(\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right)$$ ✓ 1 mark

$$\int \sin^5\theta \, d\theta = \frac{1}{16}\int \sin 5\theta - 5\sin 3\theta + 10\sin\theta \, d\theta$$

$$= \frac{1}{16}\left(-\frac{1}{5}\cos 5\theta + \frac{5}{3}\cos 3\theta - 10\cos\theta\right) + C$$ ✓ 1 mark

$$= \frac{-1}{80}\cos 5\theta + \frac{5}{48}\cos 3\theta - \frac{5}{8}\cos\theta + C$$

Question 16 (15 marks) Use a separate Writing Booklet

(a) It is given that the cubic equation $x^3 + bx^2 + cx + d = 0$ has a purely imaginary root. If the coefficients are all real numbers, show that $d = bc$ and $c > 0$. **2**

Solution

If the coefficients are all real then there is one pair of conjugate roots.
The roots are $\alpha i, -\alpha i, \beta$.

→ Some students forgot to mention

Sum of roots: $\alpha i - \alpha i + \beta = -b, \therefore \beta = -b$

Product of two roots at a time: $\alpha i(-\alpha i) + \beta\alpha i + \beta(-\alpha i) = \alpha^2 = c$

Product of three roots: $\alpha i(-\alpha i)\beta = \alpha^2\beta = -d$

By substitution:

$$\alpha^2\beta = -d$$

$$c(-b) = -d$$

$$\therefore c = \frac{-d}{-b} = \frac{d}{b} > 0$$

$$\therefore d = bc$$

– Overall, it was a well done question.

(b) A particle moves with acceleration function $\ddot{x} = 3x^2$. Initially $x = 1$ and $v = -\sqrt{2}$.

(i) Show that $v^2 = 2x^3$. **2**

(ii) Explain why the velocity can never be positive. **1**

(iii) Find the displacement-time function, and briefly describe the motion. **3**

Solution

19(a) $v^2 = 2x^3$ (b) Initially, v is negative. Since $v^2 = 2x^3$, v can only be zero at the origin. But since $\ddot{x} = 3x^2$ the acceleration at the origin would also be zero. Hence if the particle ever arrived at the origin, it would then be permanently at rest. Thus the velocity can never change from negative to positive, and must always be negative or zero. $x = \dfrac{2}{(t + \sqrt{2})^2}$. The particle starts at $x = 1$ and moves backwards towards the origin, its speed having limit zero, and position having limit the origin.

~~[illegible]~~

Part (i) – Majority of students attempted it correctly.

(ii) Few students could do it

(iii) Many students forgot the negative sign for velocity. Students need to learn how to describe a motion — poor attempt

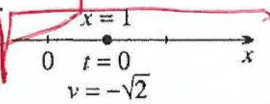
(a) Since acceleration is given as a function of displacement, we write:

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 3x^2$$
$$\tfrac{1}{2}v^2 = x^3 + \tfrac{1}{2}C, \quad \text{for some constant } C.$$
$$v^2 = 2x^3 + C, \quad \text{for some constant } C.$$

When $x = 1$, $v = -\sqrt{2}$, so $\quad 2 = 2 + C$,

so $C = 0$, and $\quad v^2 = 2x^3$.

- Many students did a good job here



→ advisable to draw.

(b) Taking square roots, $\quad v = \boxed{-}\sqrt{2}\,x^{\frac{3}{2}}$, assuming that v is never positive.

Taking reciprocals, $\quad \frac{dt}{dx} = -\tfrac{1}{2}\sqrt{2}\,x^{-\frac{3}{2}}$ → Many students forgot

$$t = \sqrt{2}\,x^{-\frac{1}{2}} + D, \quad \text{for some constant } D.$$

When $t = 0$, $x = 1$, so $\quad 0 = \sqrt{2} + D$,

so $D = -\sqrt{2}$, and $\quad t = \sqrt{2}\,x^{-\frac{1}{2}} - \sqrt{2}$

- Few students substituted incorrect conditions.

$$x^{-\frac{1}{2}} = \frac{t + \sqrt{2}}{\sqrt{2}}$$
$$x = \frac{2}{(t + \sqrt{2})^2}.$$

— Few students wrote 't' as the subject

Hence the particle begins at $x = 1$, and moves backwards towards the origin. As $t \to \infty$, its speed has limit zero, and its limiting position is $x = 0$.

(c) Initially, v is negative. Since $v^2 = 2x^3$, it follows that v can only be zero at the origin; but since $\ddot{x} = 3x^2$ the acceleration at the origin would also be zero. Hence if the particle ever arrived at the origin it would then be permanently at rest. Thus the velocity can never change from negative to positive.

(c) (i) Prove that $\sqrt{x}\left(1-\sqrt{x}\right)^{n-1} = \left(1-\sqrt{x}\right)^{n-1} - \left(1-\sqrt{x}\right)^{n}$ **1**

Solution

$$RHS = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^{n}$$
$$= (1-\sqrt{x})^{n-1}\left[1 - (1-\sqrt{x})\right] \quad 1$$
$$= (1-\sqrt{x})^{n-1}\left[1 - 1 + \sqrt{x}\right]$$
$$= \sqrt{x}\,(1-\sqrt{x})^{n-1}$$
$$= LHS.$$

- students could do it

- still for many students, they need to learn as it is a very popular style

(ii) Let $I_n = \int_0^1 \left(1-\sqrt{x}\right)^n dx$, where $n = 1, 2, 3, \ldots$ **3**

show that $I_n = \dfrac{n}{n+2} I_{n-1}$

Solution

ii) $I_n = \int_0^1 (1-\sqrt{x})^n \, dx$

let $u = (1-\sqrt{x})^n$ $\quad dv = dx$

$du = n(1-\sqrt{x})^{n-1} \times \frac{-1}{2\sqrt{x}} \, dx$ $\quad v = x.$

$\therefore I_n = uv - \int v \, du$ ①.

$= \left[x(1-\sqrt{x})^n\right]_0^1 - \int_0^1 \frac{nx(1-\sqrt{x})^{n-1}}{-2\sqrt{x}} \, dx$

$= (0-0) + \frac{n}{2}\int_0^1 \sqrt{x}(1-\sqrt{x})^{n-1} \, dx.$

(from part i) $\Rightarrow \frac{n}{2}\int_0^1 \left[(1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n\right] dx.$ ①

$= \frac{n}{2}\int_0^1 (1-\sqrt{x})^{n-1} \, dx - \frac{n}{2}\int_0^1 (1-\sqrt{x})^n \, dx$

$\therefore I_n = \frac{n}{2} I_{n-1} - \frac{n}{2} I_n$

$I_n + \frac{n}{2} I_n = \frac{n}{2} I_{n-1}$

$\left(\frac{2+n}{2}\right) I_n = \frac{n}{2} I_{n-1}$

$\therefore I_n = \frac{n}{2} \times \frac{2}{2+n} I_{n-1}$ ①.

$I_n = \frac{n}{n+2} I_{n-1}$, as required

→ many students did mention, u, u', v, v' correctly, but did a number of mistakes

Overall, students need to strengthen their reduction formula

(iii) Hence evaluate I_{2022}. **2**

Solution

iii) $I_n = \frac{n}{n+2} I_{n-1}$ $\quad (n \geq 1)$

$I_1 = \frac{1}{1+2} I_0$

$= \frac{1}{3} \times \int_0^1 (1-\sqrt{x})^0 \, dx.$

$= \frac{1}{3} \times [x]_0^1$

$= \frac{1}{3} \times 1$

$= \frac{1}{3}.$ (That is: $I_0 = 1$)

— Many students could find I_1.

$$
\begin{aligned}
I_{2022} &= \frac{2022}{2024} I_{2021} \\
&= \frac{2022}{2024} \times \frac{2021}{2023} I_{2020} \\
&= \frac{2022}{2024} \times \frac{2021}{2023} \times \frac{2020}{2022} I_{2019} \\
&= \frac{2022 \times 2021 \times 2020 \times \ldots \times 4 \times 3 \times 2}{2024 \times 2023 \times 2022 \times \ldots \times 4} I_1 \\
&= \frac{3 \times 2}{2024 \times 2023} \times \frac{1}{3} \\
&= \frac{1}{2\ 047\ 276}
\end{aligned}
$$

→ some students forgot 2

→ overall, it was okay.

End of paper